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Displaced Solar Sail Orbits: Dynamics and Applications

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We consider displaced periodic orbits at linear order in the circular restricted Earth-Moon system, where the third massless body is a solar sail. These highly non-Keplerian orbits are achieved using an extremely small sail acceleration. Prior results have been developed by using an optimal choice of the sail pitch angle, which maximizes the out-of-plane displacement. In this paper we will use solar sail propulsion to provide station-keeping at periodic orbits around the libration points using small variations in the sail's orientation. By introducing a first-order approximation, periodic orbits are derived analytically at linear order. These approximate analytical solutions are utilized in a numerical search to determine displaced periodic orbits in the full nonlinear model. Applications include continuous line-of-sight communications with the lunar poles.

I. Introduction

OVER several years, solar sailing has been studied as a novel propulsion system for space missions. Solar sail technology appears as a promising form of advanced spacecraft propulsion which can enable exciting new space-science mission concepts such as solar system exploration and deep space observation. Although solar sailing has been considered as a practical means of spacecraft propulsion only relatively recently, the fundamental ideas are by no means new (see McInnes¹ for a detailed description). A solar sail is propelled by reflecting solar photons and therefore can transform the momentum of the photons into a propulsive force. Solar sails can also be utilised for highly non-Keplerian orbits, such as orbits displaced high above the ecliptic plane (see Waters and McInnes²). Solar sails are especially suited for such non-Keplerian orbits, since they can apply a propulsive force continuously. In such trajectories, a sail can be used as a communication satellite for high latitudes. For example, the orbital plane of the sail can be displaced above the orbital plane of the Earth, so that the sail can stay fixed above the Earth at some distance, if the orbital periods are equal (see Forward³). Orbits around the collinear points of the Earth-Moon system are also of great interest because their unique positions are advantageous for several important applications in space mission design (see e.g. Szebehely⁴, Roy,⁵ Vonbun,⁶ Gómez et al.^{7,8}). In recent years several authors have tried to determine more accurate approximations (quasi-Halo orbits) of such equilibrium orbits⁹. These orbits were first studied by Farquhar¹⁰, Farquhar and Kamel⁹, Breakwell and Brown¹¹, Richardson¹², Howell^{13,14}. If an orbit maintains visibility from Earth, a spacecraft on it (near the L_2 point) can be used to provide communications between the equatorial regions of the Earth and the lunar poles. The establishment of a bridge for radio communications is crucial for forthcoming space missions, which plan to use the lunar poles. McInnes¹⁵ investigated a new family of displaced solar sail orbits near the Earth-Moon libration points. Displaced orbits have more recently been developed by Ozimek et al.¹⁶ using collocation methods. In Baoyin and McInnes^{17,18,19} and McInnes^{15,20}, the authors describe new orbits which are associated with artificial Lagrange points in the Earth-Sun system. These artificial equilibria have potential applications for future space physics and Earth observation missions. In McInnes and Simmons²¹, the authors investigate large new families of solar sail orbits, such as Sun-centered halo-type trajectories, with the sail executing a circular orbit of a chosen period above the ecliptic plane. The solar sail Earth-Moon problem differs greatly from the Earth-Sun system as the Sun-line direction varies continuously in the rotating frame and the equations

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of motion of the sail are given by a set of nonlinear, non-autonomous ordinary differential equations. We have recently investigated displaced periodic orbits at linear order in the Earth-Moon restricted three-body system, where the third massless body is a solar sail (see Simo and McInnes²²). These highly non-Keplerian orbits are achieved using an extremely small sail acceleration. It was found that for a given displacement distance above/below the Earth-Moon plane it is easier by a factor of order 3.19 to do so at L_4/L_5 compared to L_1/L_2 - ie. for a fixed sail acceleration the displacement distance at L_4/L_5 is greater than that at L_1/L_2 . In addition, displaced L_4/L_5 orbits are passively stable, making them more forgiving to sail pointing errors than highly unstable orbits at L_1/L_2 . The drawback of the new family of orbits is the increased telecommunications path-length, particularly the Moon- L_4 distance compared to the Moon- L_2 distance.

In this paper we will use solar sail propulsion to provide station-keeping at periodic orbits about the triangular libration points using small variations in the sail's orientation. We develop and implement a control methodology for maintaining the sail in the z -direction. Thus, a linear feedback controller is proposed by linearizing the z -dynamics about the triangular libration points. A simulation using this controller is then performed using constant gains.

II. System Model

In this work m_1 represents the larger primary (Earth), m_2 the smaller primary (Moon) and we will be concerned with the motion of a hybrid sail that has negligible mass. It is always assumed that the two more massive bodies are moving in circular orbits with constant angular velocity ω about their common center of mass, and the mass of the third body is too small to affect the motion of the two more massive bodies. The unit mass is taken to be the total mass of the system ($m_1 + m_2$) and the unit of length is chosen to be the constant separation R^* between m_1 and m_2 . The time unit is defined such that m_2 orbits around m_1 in time 2π . Under these considerations the masses of the primaries in the normalized system of units are $m_1 = 1 - \mu$ and $m_2 = \mu$, with $\mu = m_2/(m_1 + m_2)$ (see Figure 1). Thus, in the Earth-Moon system, the nondimensional unit acceleration is $a_{ref} = \omega^2 R^* = 2.7307 \text{ mm/s}^2$ where the Earth-Moon distance $R^* = 384400 \text{ km}$. The dashed line in Figure 1) is a line parallel to the Sun-line direction \mathbf{S} .

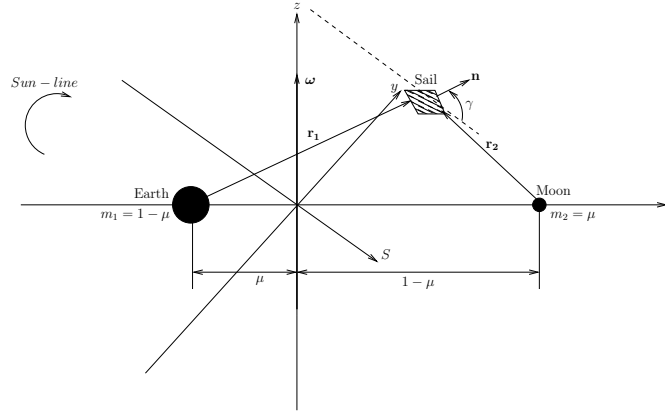


Figure 1. Schematic geometry of the Earth-Moon restricted three-body problem.

II.A. Equations of motion in presence of a solar sail

The nondimensional equation of a motion of a solar sail in the rotating frame of reference is described by

$$\frac{d^2 \mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\mathbf{r}}{dt} + \nabla U(\mathbf{r}) = \mathbf{a}_S, \quad (1)$$

where $\boldsymbol{\omega} = \omega \hat{\mathbf{z}}$ ($\hat{\mathbf{z}}$ is a unit vector pointing in the direction \mathbf{z}) is the angular velocity vector of the rotating frame and \mathbf{r} is the position vector of the sail relative to the center of mass of the two primaries. We will not consider the small annual changes in the inclination of the Sun-line with respect to the plane of the system.

The three-body gravitational potential $U(\mathbf{r})$ and the solar radiation pressure acceleration \mathbf{a}_S are defined by

$$\begin{aligned} U(\mathbf{r}) &= -\left[\frac{1}{2} |\boldsymbol{\omega} \times \mathbf{r}|^2 + \frac{1-\mu}{r_1} + \frac{\mu}{r_2} \right], \\ \mathbf{a}_S &= a_0 (\mathbf{S} \cdot \mathbf{n})^2 \mathbf{n}, \end{aligned} \quad (2)$$

where $\mu = 0.1215$ is the mass ratio for the Earth-Moon system. The sail position vectors w.r.t. m_1 and m_2 respectively (see Figure 1) are $\mathbf{r}_1 = [x + \mu, y, z]^T$ and $\mathbf{r}_2 = [x - (1 - \mu), y, z]^T$, a_0 is the magnitude of the solar radiation pressure acceleration exerted on the sail and the unit vector \mathbf{n} denotes the thrust direction. The sail is oriented such that it is always directed along the Sun-line \mathbf{S} , pitched at an angle γ to provide a constant out-of-plane force. The unit normal to the sail surface \mathbf{n} and the Sun-line direction \mathbf{S} are given by

$$\mathbf{n} = \begin{bmatrix} \cos(\gamma) \cos(\omega_\star t) & -\cos(\gamma) \sin(\omega_\star t) & \sin(\gamma) \end{bmatrix}^T, \quad (3)$$

$$\mathbf{S} = \begin{bmatrix} \cos(\omega_\star t) & -\sin(\omega_\star t) & 0 \end{bmatrix}^T, \quad (4)$$

where $\omega_\star = 0.923$ is the angular rate of the Sun-line in the corotating frame in a dimensionless synodic coordinate system.

II.B. Linearized system

We now investigate the dynamics of the sail in the neighborhood of the libration points. We denote the coordinates of the equilibrium point as $\mathbf{r}_L = (x_{L_i}, y_{L_i}, z_{L_i})$ with $i = 1, \dots, 5$. Let a small displacement in \mathbf{r}_L be $\delta \mathbf{r}$ such that $\mathbf{r} \rightarrow \mathbf{r}_L + \delta \mathbf{r}$. The equation of motion for the solar sail in the neighborhood of \mathbf{r}_L are therefore

$$\frac{d^2 \delta \mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta \mathbf{r}}{dt} + \nabla U(\mathbf{r}_L + \delta \mathbf{r}) = \mathbf{a}_S(\mathbf{r}_L + \delta \mathbf{r}). \quad (5)$$

Then, retaining only the first-order term in $\delta \mathbf{r} = [\xi, \eta, \zeta]^T$ in a Taylor-series expansion, where (ξ, η, ζ) are attached to the L_2 point as shown in Figure 1, the gradient of the potential and the acceleration can be expressed as

$$\nabla U(\mathbf{r}_L + \delta \mathbf{r}) = \nabla U(\mathbf{r}_L) + \left. \frac{\partial \nabla U(\mathbf{r})}{\partial \mathbf{r}} \right|_{\mathbf{r}=\mathbf{r}_L} \delta \mathbf{r} + O(\delta \mathbf{r}^2), \quad (6)$$

$$\mathbf{a}_S(\mathbf{r}_L + \delta \mathbf{r}) = \mathbf{a}_S(\mathbf{r}_L) + \left. \frac{\partial \mathbf{a}_S(\mathbf{r})}{\partial \mathbf{r}} \right|_{\mathbf{r}=\mathbf{r}_L} \delta \mathbf{r} + O(\delta \mathbf{r}^2). \quad (7)$$

It is assumed that $\nabla U(\mathbf{r}_L) = 0$, and the sail acceleration is constant with respect to the small displacement $\delta \mathbf{r}$, so that

$$\left. \frac{\partial \mathbf{a}_S(\mathbf{r})}{\partial \mathbf{r}} \right|_{\mathbf{r}=\mathbf{r}_L} = 0. \quad (8)$$

The linear variational system associated with the libration points at \mathbf{r}_L can be determined by substituting Eqs. (6) and (7) into (5)

$$\frac{d^2 \delta \mathbf{r}}{dt^2} + 2\boldsymbol{\omega} \times \frac{d\delta \mathbf{r}}{dt} - K \delta \mathbf{r} = \mathbf{a}_S(\mathbf{r}_L), \quad (9)$$

where the matrix K is defined as

$$K = - \left[\left. \frac{\partial \nabla U(\mathbf{r})}{\partial \mathbf{r}} \right|_{\mathbf{r}=\mathbf{r}_L} \right]. \quad (10)$$

Using the matrix notation the linearized equation about the libration point (Equation (9)) can be represented by the inhomogeneous linear system $\dot{\mathbf{X}} = A\mathbf{X} + \mathbf{b}(t)$, where the state vector $\mathbf{X} = (\delta \mathbf{r}, \delta \dot{\mathbf{r}})^T$, and $\mathbf{b}(t)$ is a 6×1 vector, which represents the solar sail acceleration.

The Jacobian matrix A has the general form

$$A = \begin{pmatrix} 0_3 & I_3 \\ K & \Omega \end{pmatrix}, \quad (11)$$

where I_3 is a identity matrix, and

$$\Omega = \begin{pmatrix} 0 & 2 & 0 \\ -2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (12)$$

For convenience the sail attitude is fixed such that the sail normal vector \mathbf{n} , points always along the direction of the Sun-line with the following constraint $\mathbf{S} \cdot \mathbf{n} \geq 0$. Its direction is described by the pitch angle γ relative to the Sun-line, which represents the sail attitude. The linearized nondimensional equations of motion relative to a collinear libration point can then be written as

$$\ddot{\xi} - 2\dot{\eta} = U_{xx}^o \xi + a_\xi, \quad (13)$$

$$\ddot{\eta} + 2\dot{\xi} = U_{yy}^o \eta + a_\eta, \quad (14)$$

$$\ddot{\zeta} = U_{zz}^o \zeta + a_\zeta, \quad (15)$$

where U_{xx}^o , U_{yy}^o , and U_{zz}^o are the partial derivatives of the gravitational potential evaluated at the collinear libration point, and the solar sail acceleration is defined in terms of three auxiliary variables a_ξ , a_η , and a_ζ

$$a_\xi = a_0 \cos(\omega_\star t) \cos^3(\gamma), \quad (16)$$

$$a_\eta = -a_0 \sin(\omega_\star t) \cos^3(\gamma), \quad (17)$$

$$a_\zeta = a_0 \cos^2(\gamma) \sin(\gamma). \quad (18)$$

In a similar fashion, recalling the linearized equations of motion obtained in the equation (9) describing the behavior of the system in the vicinity of the Lagrange points, it can be easily shown that the the linear variational equations of motion in component form at the triangular points then become

$$\ddot{\xi} - 2\dot{\eta} = U_{xx}^o \xi + U_{xy}^o \eta + a_\xi, \quad (19)$$

$$\ddot{\eta} + 2\dot{\xi} = U_{xy}^o \xi + U_{yy}^o \eta + a_\eta, \quad (20)$$

$$\ddot{\zeta} = U_{zz}^o \zeta + a_\zeta, \quad (21)$$

We will continue with the solution to the linearized equations of motion in the Earth-Moon restricted three-body problem in the next section.

III. Solution of the linearized equations of motion for the three-body model

In order to evaluate the three-body model, we will obtain a displaced periodic orbit from the linearized dynamics defined earlier. Considering the dynamics of motion near the collinear libration points, we may choose a particular periodic solution in the plane of the form (see Farquhar²³)

$$\xi(t) = \xi_0 \cos(\omega_\star t), \quad (22)$$

$$\eta(t) = \eta_0 \sin(\omega_\star t). \quad (23)$$

By inserting equations (22) and (23) in the differential equations (13-15), we obtain the linear system in ξ_0 and η_0 ,

$$\begin{cases} (U_{xx}^o - \omega_\star^2) \xi_0 - 2\omega_\star \eta_0 = a_0 \cos^3(\gamma), \\ -2\omega_\star \xi_0 + (U_{yy}^o - \omega_\star^2) \eta_0 = -a_0 \cos^3(\gamma). \end{cases} \quad (24)$$

Then the amplitudes ξ_0 and η_0 are given by

$$\xi_0 = a_0 \frac{(U_{yy}^o - \omega_\star^2 - 2\omega_\star) \cos^3(\gamma)}{(U_{xx}^o - \omega_\star^2)(U_{yy}^o - \omega_\star^2) - 4\omega_\star^2}, \quad (25)$$

$$\eta_0 = a_0 \frac{(-U_{xx}^o + \omega_\star^2 + 2\omega_\star) \cos^3(\gamma)}{(U_{xx}^o - \omega_\star^2)(U_{yy}^o - \omega_\star^2) - 4\omega_\star^2}, \quad (26)$$

and we have the equality

$$\frac{\xi_0}{\eta_0} = \frac{\omega_\star^2 + 2\omega_\star - U_{yy}^o}{-\omega_\star^2 - 2\omega_\star + U_{xx}^o}. \quad (27)$$

The trajectory will therefore be an ellipse centered on a collinear libration point. We can find the required radiation pressure acceleration by solving equation (25)

$$a_0 = \cos^{-3}(\gamma) \left[\frac{\omega_\star^4 - \omega_\star^2(U_{xx}^o + U_{yy}^o + 4) + U_{xx}^o U_{yy}^o}{U_{yy}^o - 2\omega_\star - \omega_\star^2} \right] \xi_0.$$

By applying a Laplace transform, the uncoupled out-of-plane ζ -motion defined by the equation (15) can be obtained as

$$\zeta(t) = U(t)a_0 \cos^2(\gamma) \sin(\gamma) |U_{zz}^o|^{-1} + \dot{\zeta}_0 |U_{zz}^o|^{-1/2} \sin(\omega_\zeta t) + \cos(\omega_\zeta t) [\zeta_0 - a_0 \cos^2(\gamma) \sin(\gamma) |U_{zz}^o|^{-1}] \quad (28)$$

where the nondimensional frequency ω_ζ is defined as

$$\omega_\zeta = |U_{zz}^o|^{1/2}$$

and $U(t)$ is the unit step function.

A sufficient condition for displaced orbits based on the sail pitch angle γ and the magnitude of the solar radiation pressure a_0 for fixed initial out-of-plane distance ζ_0 can be derived. Specifically for the choice of the initial data $\dot{\zeta}_0 = 0$, equation (28) can be more conveniently expressed as

$$\begin{aligned} \zeta(t) &= U(t)a_0 \cos^2(\gamma) \sin(\gamma) |U_{zz}^o|^{-1} \\ &\quad + \cos(\omega_\zeta t) [\zeta_0 - a_0 \cos^2(\gamma) \sin(\gamma) |U_{zz}^o|^{-1}]. \end{aligned} \quad (29)$$

The solution can then be made to contain only a constant displacement at an out-of-plane distance

$$\zeta_0 = a_0 \cos^2(\gamma) \sin(\gamma) |U_{zz}^o|^{-1}. \quad (30)$$

Furthermore, the out-of-plane distance can be maximized by an optimal choice of the sail pitch angle determined by

$$\begin{aligned} \left. \frac{d}{d\gamma} \cos^2(\gamma) \sin(\gamma) \right|_{\gamma=\gamma^\star} &= 0, \\ \gamma^\star &= 35.264^\circ. \end{aligned} \quad (31)$$

We now have conditions for a small displaced periodic orbit centered on the collinear libration points.

Following the idea already presented for the collinear points, since the particular solution in the plane (Eq. (22) and (23)) cannot satisfy the linear ODEs for the triangular points (Eq. (19)-(21)), the subsequent discussion is to find solutions that satisfy these differential equations.

Assume that a solution to the linearized equations of motion (19-21) is periodic of the form

$$\xi(t) = A_\xi \cos(\omega_\star t) + B_\xi \sin(\omega_\star t), \quad (32)$$

$$\eta(t) = A_\eta \cos(\omega_\star t) + B_\eta \sin(\omega_\star t), \quad (33)$$

where A_ξ , A_η , B_ξ and B_η are free parameters to be determined.

By substituting Equations (32) and (33) in the differential equations, we obtain the linear system in A_ξ , A_η , B_ξ and B_η ,

$$\begin{cases} -(\omega_\star^2 + U_{xx}^o)B_\xi + 2\omega_\star A_\eta - U_{xy}^o B_\eta = 0, \\ -U_{xy}^o A_\xi + 2\omega_\star B_\xi - (\omega_\star^2 + U_{xx}^o)A_\eta = 0, \\ -(\omega_\star^2 + U_{xx}^o)A_\xi - U_{xy}^o A_\eta - 2\omega_\star B_\eta = a_0 \cos(\gamma)^3, \\ -2\omega_\star A_\xi - U_{xy}^o B_\xi - (\omega_\star^2 + U_{yy}^o)B_\eta = -a_0 \cos(\gamma)^3. \end{cases} \quad (34)$$

Thus, the linear system may be solved to find the coefficient A_ξ , B_ξ , A_η and B_η , which will satisfy the ODEs.

For convenience, define

$$\mathbf{x} = \begin{bmatrix} A_\xi & B_\xi & A_\eta & B_\eta \end{bmatrix}^T, \quad \mathbf{A} = \begin{bmatrix} A_1 & B_1 \\ C_1 & D_1 \end{bmatrix},$$

and

$$\mathbf{b} = \begin{bmatrix} 0 & 0 & a_0 \cos^3(\gamma) & -a_0 \cos^3(\gamma) \end{bmatrix}^T,$$

where the submatrices of \mathbf{A} are given by

$$A_1 = \begin{bmatrix} 0 & -\omega_\star^2 - U_{xx}^o \\ -U_{xy}^o & 2\omega_\star \end{bmatrix}, \quad B_1 = \begin{bmatrix} 2\omega_\star & -U_{xy}^o \\ -\omega_\star^2 - U_{yy}^o & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} -\omega_\star^2 - U_{xx}^o & 0 \\ -2\omega_\star & -U_{xy}^o \end{bmatrix}, \quad D_1 = \begin{bmatrix} -U_{xy}^o & -2\omega_\star \\ 0 & -\omega_\star^2 - U_{yy}^o \end{bmatrix}.$$

We have in matrix form $\mathbf{Ax} = \mathbf{b}$, and the solution to the linear system is given by

$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b},$$

where the coefficients A_ξ , A_η , B_ξ and B_η are amplitudes that characterize the orbit. The out-of-plane dynamics (Eq. (21)) are again uncoupled and follow the same analysis as the collinear points.

This section is now concerned with the numerical computation of displaced periodic orbits around the Lagrange points in the Earth-Moon system. As has been shown, there exist displaced orbits at all Lagrange points. For example, the numerical nonlinear results for the Lagrange points L_4 (Figure 2 (a)), and L_5 (Figure 2 (b)) demonstrate, that displaced periodic orbits appear in their vicinity with a period of 28 days (synodic lunar month).

Furthermore, the numerically integrated nonlinear (solid line) equations match the linear analytic solutions (dashed line) for a small displaced orbit (Figure 3 (a), and 3 (b)).

In order to maintain the sail in the z -direction for a long period above the triangular libration points, a simple linear feedback controller will be developed.

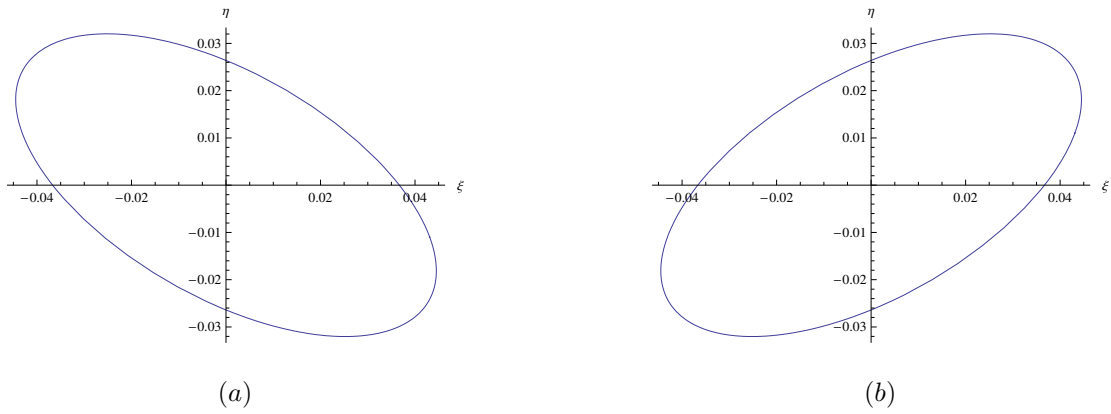


Figure 2. (a) Periodic Orbits at linear order around L_4 ; (b) Periodic Orbits at linear order around L_5 .

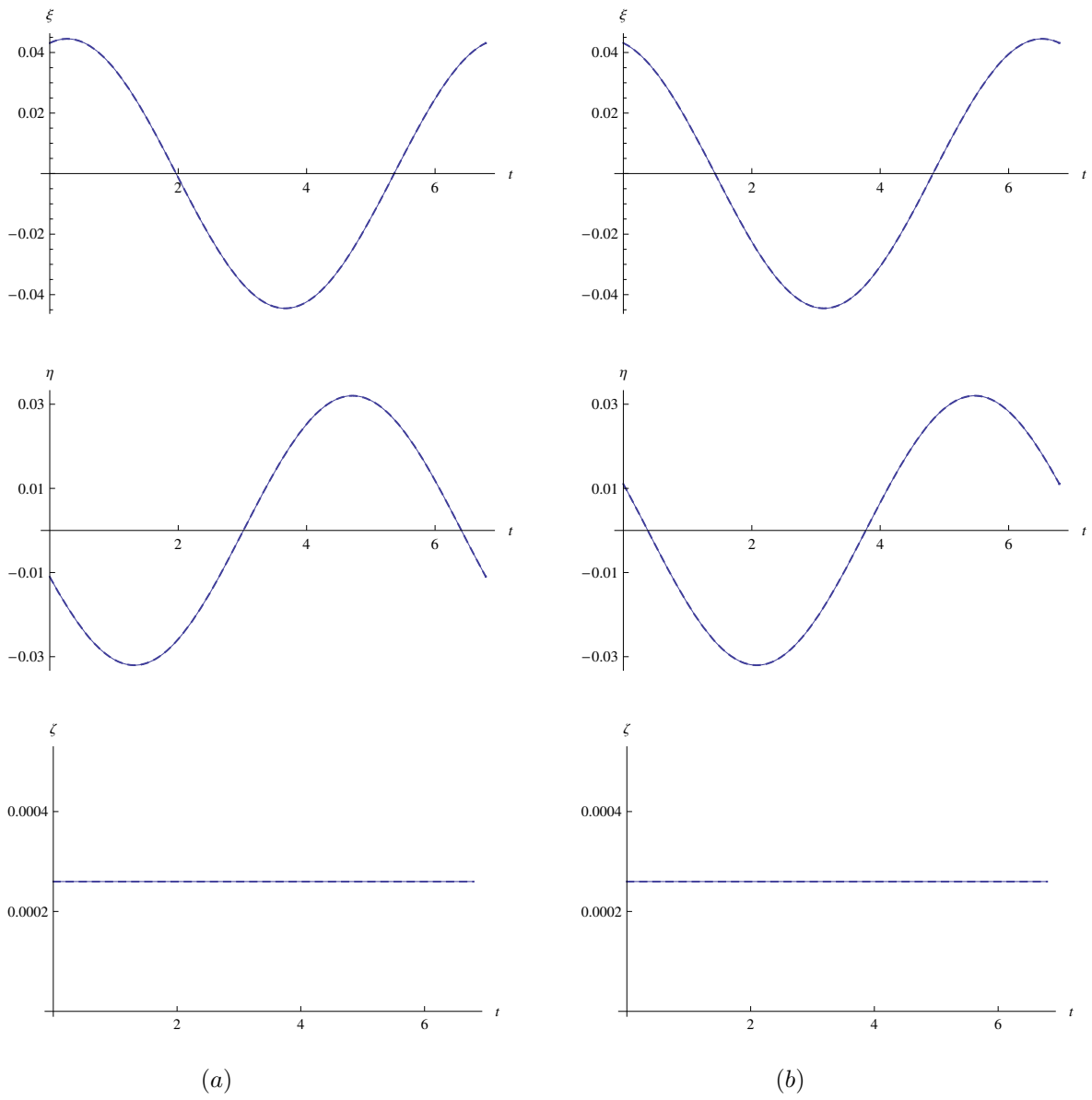


Figure 3. (a) Comparison between the analytical (dashed line) and nonlinear (solid line) results (L_4); (b) Comparison between the analytical (dashed line) and nonlinear (solid line) results (L_5).

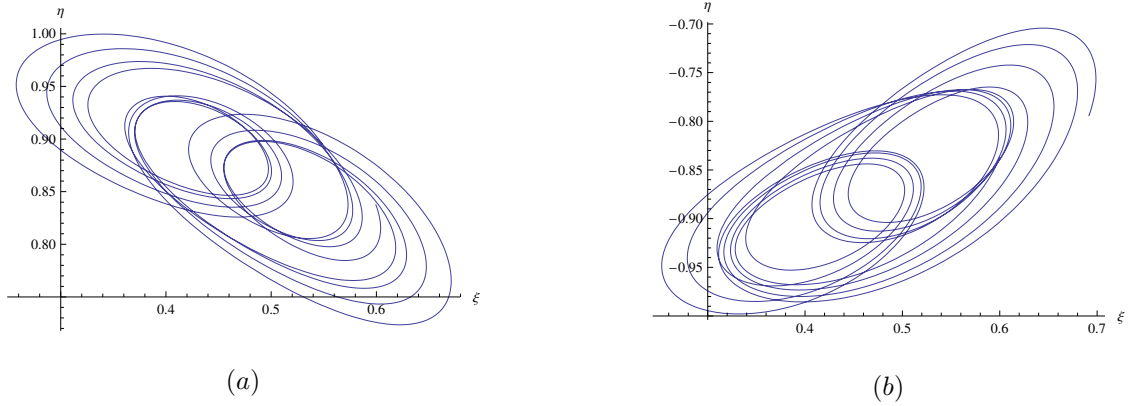


Figure 4. (a) Quasi-Periodic Orbits around L_4 ; (b) Quasi-Periodic Orbits around L_5 .

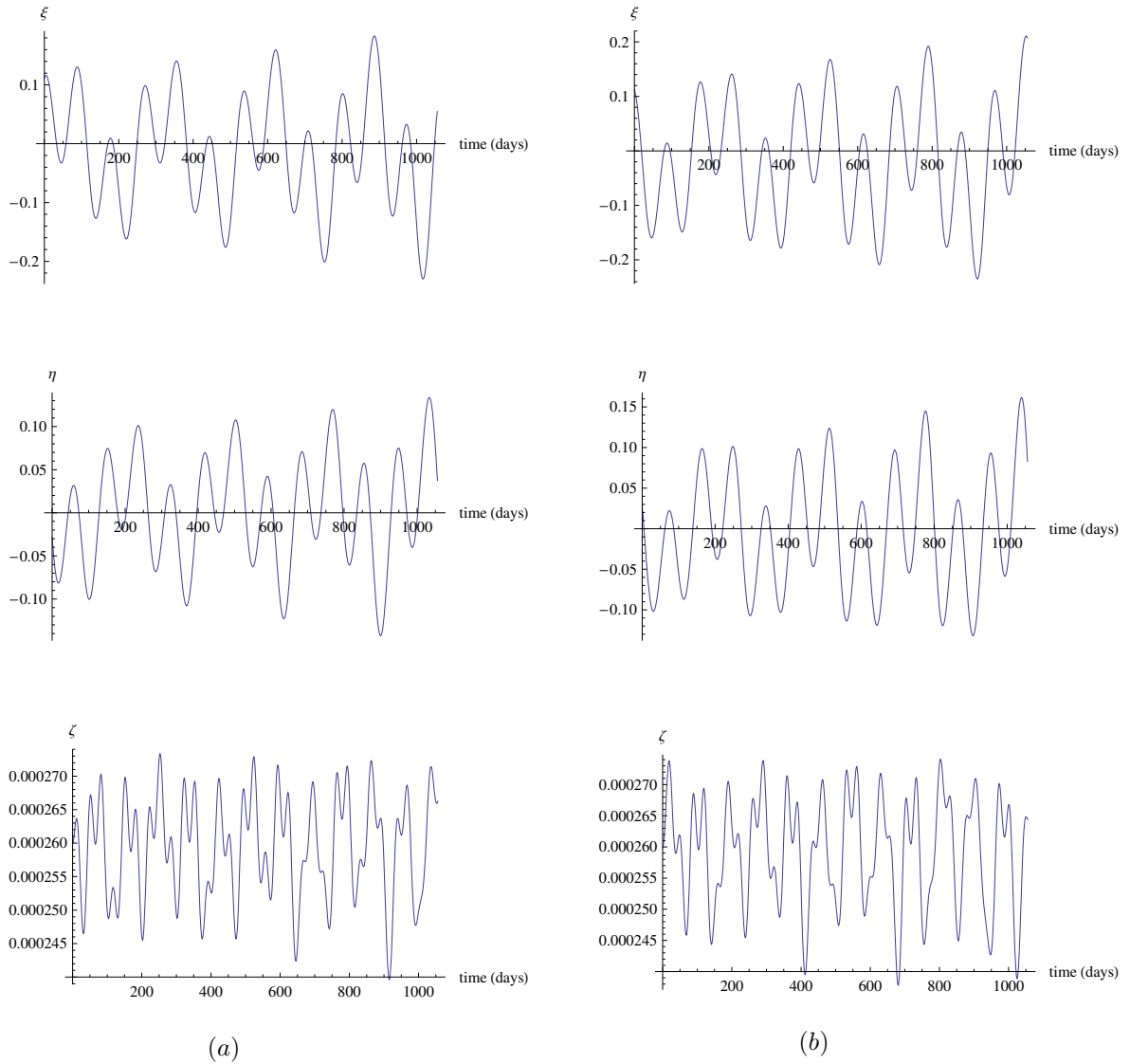


Figure 5. (a) Linear Feedback Control on Sail x, y, z -position (L_4); (b) Linear Feedback Control on Sail x, y, z -position (L_5).

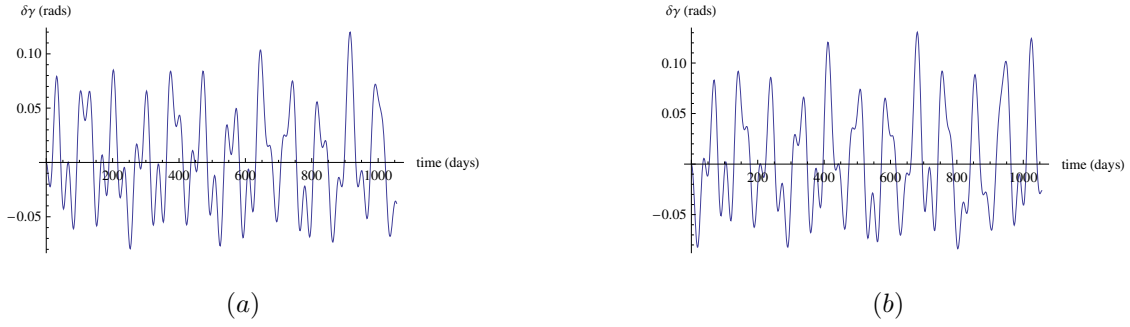


Figure 6. (a) Control History for the L_4 Quasi-Periodic Orbits; (b) Control History for the L_5 Quasi-Periodic Orbits.

IV. Control of Sail z-Position

In this section we consider the problem of maintaining a sail at a triangular libration point. To accomplish this task a simple control methodology is developed for stationing the solar sail. The control is achieved by using small variations in the sail's orientation.

Recall that the motion along the z -axis is independent of the motion in the xy -plane. Thus, a z -axis control of the sail orbit is studied. The z -position is maintained at the triangular libration points by adjusting the control angle γ in such a way that it will cancel disturbances that drive the sail away from those points.

The linear feedback controller is developed by linearizing the z -dynamics around the triangular libration points and some sail attitude γ_0 .

From the equation (21), the linearization of the uncoupled motion about γ_0 gives

$$\ddot{\zeta} = \left. \frac{\partial^2 U}{\partial z^2} \right|_o \zeta + a_0 \cos^2(\gamma_0) \sin(\gamma_0), \quad (35)$$

where the subscript o refers to the triangular libration point. If $\ddot{\zeta} = 0$, the condition for out-of-plane equilibrium is given by

$$a_0 \cos^2(\gamma_0) \sin(\gamma_0) = - \left. \frac{\partial^2 U}{\partial z^2} \right|_o \zeta_0. \quad (36)$$

Now let

$$\zeta = \zeta_0 + \delta\zeta, \quad (37)$$

$$\gamma = \gamma_0 + \delta\gamma. \quad (38)$$

By making use of Eqs. (37) and (38), the uncoupled motion (Eq. (21)) can be stated as

$$\frac{d^2}{dt^2}(\zeta_0 + \delta\zeta) = \left. \frac{\partial^2 U}{\partial z^2} \right|_o (\zeta_0 + \delta\zeta) + a_0 \cos^2(\gamma_0) \sin(\gamma_0) + a_0 (\cos^3(\gamma_0) - 2 \cos(\gamma_0) \sin^2(\gamma_0)) \delta\gamma, \quad (39)$$

Applying Eq. (36), then Eq. (39) can now be rewritten as

$$\delta\ddot{\zeta} = \left. \frac{\partial^2 U}{\partial z^2} \right|_o \delta\zeta + a_0 (\cos^3(\gamma_0) - 2 \cos(\gamma_0) \sin^2(\gamma_0)) \delta\gamma, \quad (40)$$

where $\gamma_0 \neq 35.264^\circ$.

By setting

$$A = \left. \frac{\partial^2 U}{\partial z^2} \right|_o, \quad (41)$$

$$B = a_0 (\cos^3(\gamma_0) - 2 \cos(\gamma_0) \sin^2(\gamma_0)) \delta\gamma, \quad (42)$$

Eq. (40) can also be rearranged as

$$\delta\ddot{\zeta} = A \cdot \delta\zeta + B \cdot \delta\gamma. \quad (43)$$

Now it is possible to design the linear feedback controller of the form

$$\delta\gamma = C \cdot \delta\zeta + D \cdot \delta\dot{\zeta}, \quad (44)$$

where C and D are the PD-controller gains. Thus, the PD-controller will maintain the sail at a fixed displacement above the triangular libration points.

Substituting Eq. (44) into Eq. (43), we obtain

$$\begin{aligned} \delta\ddot{\zeta} &= A \cdot \delta\zeta + B(C \cdot \delta\zeta + D \cdot \delta\dot{\zeta}), \\ &= (A + BC) \cdot \delta\zeta + D \cdot \delta\dot{\zeta}, \end{aligned} \quad (45)$$

and

$$\delta\ddot{\zeta} - (A + BC) \cdot \delta\zeta - D \cdot \delta\dot{\zeta} = 0, \quad (46)$$

where D the damping coefficient is chosen such that the system converges as a critically damped system.

Figure 4(a) (resp. 4(b)) shows the sail's trajectory around L_4 (resp. L_5) using the linear feedback controller on the nonlinear system. Figure 5(a) (resp. 5(b)) shows the simulation results for L_4 (resp. L_5) using this controller on sail x, y, z -position. The control angle history for the L_4 (resp. L_5) quasi-periodic orbits is shown in Figure 6 (a) (resp. 6 (b)). It should be noted that while the z displacement is almost constant, in-plane dynamics are excited.

V. Conclusion

Summarizing, it can be stated that, following numerical computations around the triangular libration points in the Earth-Moon system, the linear feedback controller approach based upon the z -dynamics is successful in maintaining a sail at a specific constant attitude.

As already mentioned, a sufficient condition for displaced periodic orbits based on the sail pitch angle and the magnitude of the solar radiation pressure for fixed initial out-of-plane distance has been derived.

A particular use of such orbits include continuous communications between the equatorial regions of the Earth and the lunar poles.

Acknowledgments

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